ZFCρ Thermodynamics Paper III: Canonical f/r Extraction and Regime Classification of the Fluctuation Absorption Rate η

Han Qin

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Abstract

Nonequilibrium statistical mechanics lacks a scalar parameter that is simultaneously dimensionless, approximately cross-scale stable, dynamically endogenous, and operationally measurable. This paper introduces the convention-corrected coupling κ, which quantifies the compensatory coupling between a driving channel f and a restoring channel r in dissipative two-channel dynamics, independent of sign conventions. Four regimes are classified: null ($\kappa \approx 0$, f and r uncoupled), absorptive ($0 < \kappa < 1$, partial absorption), overscreening ($\kappa > 1$, over-compensation), and anti-compensatory ($\kappa < 0$, co-amplification).

We define the algebraic continuation $\bar{\eta} := 1 - \kappa$, reportable in all regimes. The name "fluctuation absorption rate η" is reserved for the absorptive regime, where $\eta = \bar{\eta} = 1 - \kappa$ carries a clear physical meaning: the fraction of driving fluctuations not offset by the restoring response in each discrete step.

The concept of an admissible decomposition family is introduced: η does not claim invariance under arbitrary basis rotations, but seeks identifiability, reproducibility, and cross-scale testability within physically interpretable decomposition families.

A τ-selection taxonomy distinguishes two system classes. In channel-independent systems (e.g. Ornstein–Uhlenbeck processes, Lindley queues), f and r are independent stochastic processes; $\bar{\eta}(\tau)$ may exhibit a null corner at small τ. In state-coupled systems (e.g. Brusselator chemical oscillations, Schlögl bistable model), f and r share state variables; $\bar{\eta}(\tau)$ enters a plateau directly at small τ without an under-resolved region.

Three benchmark systems form a discriminating triad. The Ornstein–Uhlenbeck process ($\kappa \approx 0$, $\bar{\eta} \approx 1$) serves as a null test. The Brusselator ($\kappa \approx 0.8$, $\eta \approx 0.19$–0.20) exhibits an absorptive plateau stable across noise intensities $\sigma/x^* \leqslant 1$. The Schlögl model ($\kappa > 1$, $\bar{\eta} < 0$ throughout) exhibits overscreening.

The Brusselator's $\eta \approx 0.20$ coincides with the value observed in ZFCρ prime-step recursion, a mathematically and physically unrelated system. Parameter-space scanning reveals that η is not a constant but a continuous dynamical indicator: under moderate noise ($\sigma/x^* \leqslant 1$), η ranges over [0.14, 0.34] across the oscillatory regime with mean 0.20, increasing monotonically with distance from the Hopf bifurcation.

A candidate unifying ansatz is reported: $\eta \approx 0.20$ can be factored as the composite dissipation $1 - C(1)^{m_{eff}}$, where $C(1)$ is the lag-1 autocorrelation (per-step retention) and m_{eff} the effective decay depth. In ZFCρ, $C(1) = 0.96$, $m_{eff} = 5.47$; in the Brusselator, $C(1) = 0.98$, $m_{eff} = 11.3$. The individual parameters differ but the composite converges. We note that m_{eff} in the Brusselator is inferred from η and $C(1)$ rather than independently measured; this factorization is therefore suggestive rather than conclusive. Whether $\eta \approx 0.20$ represents a fixed point of the composite dissipation function in state-coupled systems is a high-priority open problem.

Four falsifiable predictions are stated. The strongest is experimental: in Belousov–Zhabotinsky reaction data, if synthesis and degradation rates can be independently measured under conditions where noise does not dominate the deterministic coupling, η should lie in $[0.14, 0.34]$.

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1. Introduction: a four-criterion gap in nonequilibrium parameters

1.1 The problem

Equilibrium statistical mechanics has a complete toolkit. Partition functions bridge micro and macro scales exactly; free-energy minimisation gives equilibrium criteria; the fluctuation-dissipation theorem yields transport coefficients in the linear-response regime.

Away from equilibrium, most of these tools fail. The most important advances of the past thirty years are the Jarzynski equality and the Crooks fluctuation theorem [1, 2], which give exact identities relating work and free-energy differences arbitrarily far from equilibrium. But identities are not dynamical theories: they say "on average, the free-energy difference is correct," not "how much leaks per step."

Many quantities describe nonequilibrium behaviour. When we ask which of them simultaneously satisfy four operational criteria, a gap appears.

The four criteria are: dimensionless (cross-system comparability), approximately cross-scale stable (similar values at different coarse-graining scales), dynamically endogenous (arising from the system's drive–response structure, not an externally imposed statistic), and operationally measurable (with an explicit extraction protocol and error estimate).

1.2 Existing quantities

The entropy production rate $\sigma = dS_i/dt$ is the closest candidate. It is non-negative, dynamically endogenous, and measurable. But it carries dimensions ($J\,K^{-1}\,s^{-1}$) and depends on system size; σ for a bacterium and σ for a star are not comparable.

The Jarzynski dissipated work $W_{diss} = W - \Delta F$ can be rendered dimensionless under special conditions, but describes a single-trajectory path quantity rather than an intrinsic dynamical parameter.

The effective temperature T_eff from FDR violation characterises distance from equilibrium, but depends on which observable is chosen [3] and is not an intrinsic system property.

The Lyapunov exponent λ is dimensionless and cross-scale stable, but measures orbital instability (chaos), not dissipation. Conservative Hamiltonian systems can have positive Lyapunov exponents.

The Hurst exponent H is dimensionless and cross-scale stable, but is a statistical descriptor (time-series self-similarity), not a product of dynamical mechanism.

The thermodynamic uncertainty relation (TUR) provides a lower bound on current fluctuations [4] and is among the sharpest results in nonequilibrium physics. But TUR constrains the precision–cost trade-off of a single current, not the coupling between two independently observable processes.

We also checked the Fano factor, the Kullback–Leibler divergence rate (relative entropy production rate), excess entropy production in the sense of Maes [5], and dynamical activity in the sense of Lecomte, Appert-Rolland, and van Wijland [6]. Each satisfies two or three of the four criteria, but none satisfies all four.

Quantity	**Dimensionless**	**Cross-scale stable**	**Dynamically endogenous**	**Operationally measurable**
σ (entropy production rate)	×	×	✓	✓
W_diss (Jarzynski)	×	×	×	✓
T_eff (FDR violation)	×	×	×	×
λ (Lyapunov)	✓	✓	✓*	×
H (Hurst)	✓	✓	×	×
TUR Q-value	✓	○	○	✓
η (this paper)	✓	✓	✓	✓

Note: ✓* indicates that the Lyapunov exponent is a geometric rather than a thermodynamic quantity. ○ indicates partial satisfaction. This table does not claim η is unique; the design aim is to combine all four desiderata in one measurable scalar.

1.3 Contributions

This paper proposes the fluctuation absorption rate η as a candidate to fill the gap. η is defined via decomposing system dynamics into a driving channel f (forward) and a restoring channel r (reset). η measures the fraction of f-fluctuations that r fails to offset.

The concept draws on the Le Chatelier intuition of compensatory response: a system responds to perturbation by partially counteracting it [7]. η quantifies the "partially." But η is not a near-equilibrium linear-response coefficient. It requires neither near-equilibrium conditions, nor linearisation, nor Onsager reciprocal relations. η is a far-from-equilibrium operationalisation of the Le Chatelier spirit, not a quantitative version of the Le Chatelier principle itself.

The paper has four contributions: (1) convention-corrected coupling κ and four-regime classification; (2) admissible decomposition family; (3) τ-selection taxonomy (channel-independent vs state-coupled systems); (4) three-system benchmark panel (OU / Brusselator / Schlögl).

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2. Convention-corrected coupling κ and regime classification

2.1 Sign conventions and the definition of κ

Consider a dissipative system whose observable X evolves in discrete steps as

$$\Delta X_n = \Delta f_n + s_r \cdot \Delta r_n + \xi_n$$

where Δf_n is the contribution of the driving channel (which, acting alone, would push the system away from its steady state), Δr_n is the restoring channel's contribution, $s_r \in \{+1, -1\}$ is a sign-convention flag, and ξ_n is external noise.

In the drive–restoration convention ($\Delta X = \Delta f - \Delta r$), $s_r = -1$. If the negative sign has been absorbed into Δr ($\Delta X = \Delta f + \Delta r$), then $s_r = +1$.

This sign difference flips the sign of $\mathrm{Cov}(\Delta f, \Delta r)$, so the same physical system may be reported with different $\bar{\eta}$ values depending solely on the convention chosen.

To eliminate this dependence, we define the convention-corrected coupling:

$$\kappa := -s_r \cdot \mathrm{Cov}(\Delta f, \Delta r) / \mathrm{Var}(\Delta f)$$

Physically, κ is the fraction of f-fluctuations offset by the r-response after removing the sign-convention ambiguity.

Throughout this paper, we adopt the f − r convention ($\Delta X = \Delta f - \Delta r$), so $s_r = -1$. Under this convention, $\mathrm{Cov}(\Delta f, \Delta r) > 0$ indicates compensation (f rises → r follows → net effect is damped), and $\kappa = \mathrm{Cov} / \mathrm{Var}(\Delta f)$.

2.2 Four regimes

The value of κ partitions systems into four classes.

Null regime ($\kappa \approx 0$). f- and r-fluctuations are nearly uncorrelated. The restoring channel has negligible predictive power over the driving channel. The Ornstein–Uhlenbeck process falls here when decomposed into noise kicks and restoring force (at small τ, the external kick and the current restoring channel are approximately independent).

Absorptive regime ($0 < \kappa < 1$). The restoring channel offsets part, but not all, of the driving fluctuations. The fluctuation absorption rate $\eta := 1 - \kappa$ has a clear physical meaning in this regime: it is the fraction of driving fluctuations left unabsorbed per step.

Overscreening regime ($\kappa > 1$). The restoring response exceeds the driving fluctuations. This is mathematically legitimate: Cauchy–Schwarz only bounds $|\mathrm{Cov}| \leq \sqrt{(\mathrm{Var}(f)\ \mathrm{Var}(r))}$, so the ratio can exceed 1 when $\mathrm{Var}(r) > \mathrm{Var}(f)$. Physically, the restoring force overshoots. The Schlögl model lies here under its natural decomposition.

Anti-compensatory regime ($\kappa < 0$). The corrected coupling is negative, meaning f and r co-amplify rather than compensate. This typically signals a misidentified decomposition or a far-from-steady-state transient.

2.3 $\bar{\eta}$ and the three-family definition of η

The algebraic continuation $\bar{\eta} := 1 - \kappa$ is computable and reportable in all regimes. The name "fluctuation absorption rate η" is reserved for the absorptive regime. In the null regime one reports $\kappa \approx 0$ or $\bar{\eta} \approx 1$; in the overscreening regime, $\kappa > 1$ or $\bar{\eta} < 0$.

Within the absorptive regime, the primary quantity is

$\eta_{r\leftarrow f} := 1 - \kappa = 1 + s_r \cdot \mathrm{Cov}(\Delta f, \Delta r) / \mathrm{Var}(\Delta f)$

This is the r-upon-f absorption residual. $\eta_{r\leftarrow f} = 0$ means perfect absorption; $\eta_{r\leftarrow f} = 1$ means no absorption.

The reverse diagnostic

$\eta_{f\leftarrow r} := 1 + s_r \cdot \mathrm{Cov}(\Delta f, \Delta r) / \mathrm{Var}(\Delta r)$

measures f's offsetting of r. In most physical systems $\eta_{r\leftarrow f} \neq \eta_{f\leftarrow r}$ because $\mathrm{Var}(f) \neq \mathrm{Var}(r)$. The asymmetry itself carries information: a large gap indicates a power imbalance between the two channels.

The symmetric coupling residual

$\eta_{sym} := 1 - |\mathrm{Corr}(\Delta f, \Delta r)|$

is independent of which channel serves as denominator, measuring total coupling residual. It is useful when the drive/restore assignment is uncertain.

2.4 Compatibility with Thermo I

Thermo I (DOI: 10.5281/zenodo.19310282) defined $\eta = 1 - |\mathrm{Cov}(\Delta f, \Delta r)| / \mathrm{Var}(\Delta f)$ using absolute value. In Thermo I's DP recursion and Lindley-queue verification, Cov was consistently negative (under the f + r convention), so |Cov| = −Cov and the Thermo I η coincides numerically with the present $\eta_{r\leftarrow f}$. The κ framework generalises, rather than replaces, the Thermo I definition.

2.5 Protocol A: canonical extraction in six steps

Step 1 (Choose observable). Select the system observable $X(t)$.

Step 2 (Declare f/r decomposition and convention). Within the admissible family (§3), choose an f/r decomposition and declare s_r. If the dynamical equations are known, read f and r directly from the equations (§3.4). If not, use Protocol B (Appendix A) as a heuristic proxy.

Step 3 (Scan τ). Generate a candidate list of τ values (e.g. logarithmically spaced). For each τ, integrate $f(t)$ and $r(t)$ over τ-windows to obtain the Δf_n and Δr_n sequences. The operational workflow is sequential: first propose a candidate decomposition, then scan τ, then compare plateau widths and causal interpretability.

Step 4 (Compute). For each τ, compute κ, $\bar{\eta}$, $\eta_{r\leftarrow f}$, $\eta_{f\leftarrow r}$, η_{sym}. Report the raw sign of Cov.

Step 5 (Classify regime). Determine regime from κ. Interpret $\bar{\eta}$ as a fluctuation absorption rate only in the absorptive regime.

Step 6 (Report stability). At canonical τ (§4.2), compute $\bar{\eta}(\tau)$, $\bar{\eta}(2\tau)$, $\bar{\eta}(4\tau)$; report bootstrap 95% CIs and stability grade (§4.4).

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3. Admissible decomposition family

3.1 $\bar{\eta}$ is not a coordinate invariant

Given a system's f and r channels, any linear rotation $f' = \cos\theta \cdot f + \sin\theta \cdot r$, $r' = -\sin\theta \cdot f + \cos\theta \cdot r$ defines a new pair of "channels." Computing $\bar{\eta}$ at different θ generally yields different values.

Numerical experiments on the Brusselator confirm this (§5.2.3). The natural decomposition ($\theta = 0°$) gives $\eta \approx 0.19$. Rotation to $\theta = 90°$ gives $\bar{\eta} \approx 0.006$. Some angles produce $\bar{\eta} < 0$.

3.2 A protocol-defined quantity

η does not claim global uniqueness under arbitrary basis rotations. It is a protocol-defined quantity whose value depends on three choices: (1) f/r decomposition; (2) time scale τ; (3) statistical estimation method.

This parallels many observables in experimental physics. A crystal's elastic modulus depends on the chosen stress direction. A fluid's viscosity depends on shear rate. These quantities are not "arbitrary": they are reproducible under a given protocol, comparable across laboratories, and informative about the material. But they do depend on the measurement protocol.

η is exactly the same. We do not solve the uniqueness problem for arbitrary decompositions; we solve the identifiability problem for natural, physically interpretable decompositions.

3.3 Definition of the admissible family

An f/r decomposition is admissible if it satisfies three conditions:

(i) Physical interpretability. f and r can each be independently assigned a physical meaning. In a chemical reaction, f = synthesis rate and r = degradation rate. In a queueing system, f = arrivals and r = service. Pure mathematical rotations (e.g. the $\theta = 45°$ mixed basis) typically fail this condition.

(ii) Causal-directional consistency. In the $\kappa > 0$ regime, the roles of f and r are consistent with their dynamical identities. For channel-independent systems, time-lag analysis should show f does not systematically lag behind r. For state-coupled systems (f and r share state variables), the coupling is typically zero-lag structural coupling; the causal direction is then defined by the equation's drive/restore roles (e.g. synthesis = f, degradation = r), not by temporal lead–lag.

(iii) τ-stability. In the neighbourhood of canonical τ (§4), $\bar{\eta}(\tau)$ lies on a stable plateau (meeting the cross-scale stability criteria of §4.4).

Within the admissible family, different decompositions may give different η values. In practice, the physically natural decomposition is usually unique or nearly so. In the Brusselator, $f = a + x^2y$ (synthesis) and $r = (b+1)x$ (degradation) is the only natural decomposition. In Lindley queues, f = inter-arrival time and r = service time is likewise unique.

3.4 Reading from equations vs inferring from data

When the dynamical equations are known (model-known case), f and r can be read directly from the equations. This is the simplest and most reliable case.

When only observational data are available (model-unknown case), f and r must be inferred from data. This is harder and generally requires additional assumptions. Protocol B (Appendix A) provides one heuristic approach, with lower reliability than the model-known case.

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4. τ-selection taxonomy

4.1 Two system classes

Numerical experiments reveal a previously undiscussed fundamental distinction.

Channel-independent systems. f and r are separate stochastic processes that influence each other indirectly through the system state but do not share dynamical variables. Lindley queues (f = inter-arrival times, r = service times, mutually independent) and the Ornstein–Uhlenbeck process (at small τ, external noise kicks are approximately independent of the current restoring channel) belong here.

In channel-independent systems, f–r covariance at very short τ is dominated by external noise. As $\tau \to 0$, $\mathrm{Cov}(\Delta f, \Delta r)$ may approach zero (no instantaneous causal link), pushing $\bar{\eta}$ toward 1. This is the under-resolved region.

State-coupled systems. f and r share state variables; each channel's instantaneous value depends directly on the same variable the other channel operates on. The Brusselator ($f = a + x^2 y$, $r = (b+1)x$, both functions of x) and the Schlögl model ($f = \mu x + a$, $r = x^3$, also sharing x) belong here.

In state-coupled systems, the f–r coupling is a deterministic structural product that cannot be diluted by stochastic noise. Even at very short τ, $\mathrm{Cov}(\Delta f, \Delta r)$ is dominated by the deterministic dependence of f and r on shared state variables. $\bar{\eta}(\tau)$ enters a plateau directly at small τ without passing through an under-resolved region.

4.2 Definition of canonical τ

The canonical τ is the smallest time scale simultaneously satisfying two conditions:

Causal interpretability. For channel-independent systems, time-lag analysis at this τ shows that f does not systematically lag behind r, and the sign of κ is stable over consecutive steps. For state-coupled systems, zero-lag structural coupling is permitted; the requirement is that the f/r decomposition is consistent with the equation-level drive/restore roles.

Plateau stability. $\bar{\eta}(\tau)$, $\bar{\eta}(2\tau)$, $\bar{\eta}(4\tau)$ satisfy the cross-scale stability criteria (§4.4).

In channel-independent systems, canonical τ is typically near the relaxation time $\tau\_relax = 1/|\lambda_1|$ (λ_1 being the dominant eigenvalue of the linearised Jacobian). Below this lies the under-resolved region; above it, the over-smoothed region.

In state-coupled systems, canonical τ can be far smaller than τ_relax. The Brusselator has τ_relax ≈ 2 (from eigenvalues $0.5 \pm 0.866i$), but the η plateau begins at $\tau \approx 0.01$. Canonical τ is set by numerical resolution and sample size, not by the system's intrinsic dynamics.

4.3 Two η̄(τ) morphologies

Channel-independent systems exhibit a three-region $\bar{\eta}(\tau)$ structure: under-resolved region ($\bar{\eta}$ elevated, tending toward 1) → canonical plateau → over-smoothed region ($\bar{\eta}$ approaches an asymptotic value).

State-coupled systems exhibit a two-region structure: plateau (beginning at very small τ) → monotone decline (tending toward 0 or negative values at large τ). The Brusselator at $\sigma = 0.1$ shows $\bar{\eta}$ declining from 0.192 at $\tau = 0.01$ to -0.05 at $\tau = 100$. No under-resolved region exists.

4.4 Two-layer cross-scale stability criteria

Statistical layer. Compute $\bar{\eta}$ and its bootstrap 95% confidence interval at τ, 2τ, and 4τ. If the three CIs overlap, the statistical layer is passed.

Effect layer. When $|\bar{\eta}(\tau)| \geqslant 0.1$, compute the maximum relative drift $\Delta\_rel = \max|\bar{\eta}(k\tau) - \bar{\eta}(\tau)| / |\bar{\eta}(\tau)|$. When $|\bar{\eta}(\tau)| < 0.1$ (near-null or regime boundaries), use absolute drift $\Delta\_abs = \max|\bar{\eta}(k\tau) - \bar{\eta}(\tau)|$. Grades:

- $\Delta < 10\%$ (relative) or < 0.01 (absolute): rigid
- 10%–15% or 0.01–0.015: stable
- 15%–20% or 0.015–0.02: provisionally stable
- Beyond: scale-drifting

Report the stability grade alongside every $\bar{\eta}$ value.

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5. Benchmark panel

5.1 Experimental design

Three systems are tested, all with known dynamical equations (model-known case); f/r decompositions are read directly from the equations.

System	Dynamics	f	r	Expected regime
Ornstein-Uhlenbeck	$dx = -\gamma x\, dt + \sigma\, dW$	$\sigma\xi$ (noise kick)	γx (restoring force)	null
Brusselator	$dx/dt = a + x^2y - (b+1)x$	$a + x^2y$ (synthesis)	$(b+1)x$ (degradation)	absorptive
Schlögl	$dx/dt = -x^3 + \mu x + a + \sigma\xi$	$\mu x + a$ (linear drive)	x^3 (cubic restoring)	overscreening

The Brusselator is the most classical theoretical model of chemical oscillation (Prigogine 1968). The experimentally most studied system is the Belousov–Zhabotinsky reaction and its Oregonator reduction.

All systems are discretised via Euler–Maruyama. Brusselator: $dt = 0.001$, $T = 5000$ (5×10^6 steps); OU: $dt = 0.001$, $T = 5000$; Schlögl: $dt = 0.001$, $T = 2000$. The first 10% of data is discarded as burn-in. Statistical errors are estimated by block bootstrap (block size: 50 τ-windows) on the Δf and Δr

sequences. Multiple noise intensities are tested per system. Sign convention: f − r ($\Delta X = \Delta f - \Delta r$), s_r = −1.

5.2 Results

5.2.1 Ornstein–Uhlenbeck: null test

Parameters: $\gamma \in \{0.5, 1.0, 5.0\}$, $\sigma = 1.0$, $T = 5000$.

γ	**κ(τ = 0.01)**	**η̄(τ = 0.01)**	**η̄(τ = 0.1)**	**η̄(τ = 1.0)**
0.5	0.000	1.000	0.996	0.975
1.0	0.000	1.000	0.994	0.941
5.0	0.002	0.998	0.998	0.934

$\kappa \approx 0$ and $\bar{\eta} \approx 1$ throughout. The protocol correctly identifies null regime. This is the zero-check: had κ reported ≈ 0.8 when f and r are independent, the protocol would have a systematic bias. The OU result rules this out.

5.2.2 Brusselator: absorptive plateau

Parameters: $a = 1$, $b = 3$ (oscillatory regime, $b > 1 + a^2 = 2$), $\sigma \in \{0.1, 0.5, 1.0, 3.0\}$, $T = 500$–5000.

σ	**η(τ = 0.01)**	**η(τ = 0.1)**	**η(τ = 1.0)**	**η(τ = 10.0)**
0.1	0.192	0.189	0.109	0.080
0.5	0.174	0.165	0.058	−0.048
1.0	0.200	0.177	0.042	−0.029
3.0	0.432	0.241	0.017	−0.026

Three key observations:

First, for $\sigma = 0.1$ to 1.0, small-τ η is stable at 0.17–0.20. At $\sigma = 1.0$, $\eta(\tau = 0.001) = 0.200$. The value derives from the deterministic coupling structure of f and r (both functions of x), not from noise statistics.

Second, $\bar{\eta}(\tau)$ decreases monotonically from the small-τ plateau toward 0 (and eventually negative values, entering the overscreening regime) at large τ. There is no under-resolved region, because $f = a + x^2y$ and $r = (b+1)x$ share the state variable x; coupling persists at all time scales.

Third, at $\sigma = 3.0$ (extreme noise, $\sigma/x^* = 3.0$), small-τ η rises to 0.43. Noise begins to disrupt the deterministic f–r coupling. Even so, $\eta(\tau = 0.1) = 0.24$ remains in the absorptive regime.

The asymmetry between $\eta_{r\leftarrow f}$ and $\eta_{f\leftarrow r}$ is large: at $\tau = 0.01$, $\eta_{r\leftarrow f} \approx 0.192$ while $\eta_{f\leftarrow r} \approx 0.002$. From the definitions, $\kappa_{r\leftarrow f} = 0.808$ and $\kappa_{f\leftarrow r} = 0.998$, whose ratio gives $\mathrm{Var}(r)/\mathrm{Var}(f) = \kappa_{r\leftarrow f}/\kappa_{f\leftarrow r} \approx 0.81$. That is, Var(f) is slightly larger than Var(r). $\kappa_{f\leftarrow r}$ is close to 1 because Cov and Var(r) are numerically close (both governed by the oscillation amplitude of x), making $\mathrm{Cov}/\mathrm{Var}(r) \approx 1$. Meanwhile Var(f) exceeds Var(r) slightly (the x^2y term in $a + x^2y$ generates additional variance at oscillation peaks), so $\mathrm{Cov}/\mathrm{Var}(f) \approx 0.81$. Physically: from r's viewpoint, f nearly perfectly predicts r ($\kappa_{f\leftarrow r} \approx 1$); from f's viewpoint, r offsets only about 80% of f's fluctuations ($\kappa_{r\leftarrow f} \approx 0.8$). The asymmetry reflects the variance ratio of the two channels.

5.2.3 Brusselator: f/r rotation experiment

This section performs a basis-sensitivity scan at the comparison scale $\tau = \tau_{relax} \approx 2.0$, chosen to ensure adequate sample sizes for all rotation angles. It does not report the natural decomposition's canonical small-τ plateau value (which is $\eta \approx 0.19$–0.20, see §5.2.2).

At $\theta = 0°$ (natural), $\bar{\eta} = 0.062$. At $\theta = 45°$, $\bar{\eta} = 0.985$. At $\theta = 90°$ (f and r swapped), $\bar{\eta} = 0.006$. At $\theta \approx 142.5°$, $\bar{\eta}$ reaches its minimum at about -2.7 (overscreening).

This confirms §3.1: $\bar{\eta}$ is not a rotation invariant. The natural decomposition (synthesis/degradation) gives a value in the absorptive regime. Non-physical rotations produce either near-1 (null-like) or negative (overscreening-like) values.

5.2.4 Schlögl: overscreening

Parameters: $\mu = 1.0$, $a = 0.5$ (monostable) and $\mu = 2.5$, $a = 0.0$ (near-bifurcation), $\sigma \in \{0.1, 0.5, 1.0\}$, $T = 2000$.

$\bar{\eta}$ is negative for all parameter combinations and all τ.

(μ, a)	**σ**	**$\bar{\eta}(\tau = 0.01)$**	**$\bar{\eta}(\tau = 1.0)$**	**$\bar{\eta}(\tau = 10.0)$**
(1.0, 0.5)	0.1	−3.25	−3.24	−3.23
(1.0, 0.5)	0.5	−2.27	−1.84	−0.96
(1.0, 0.5)	1.0	−0.95	−0.74	−0.63
(2.5, 0.0)	0.1	−2.00	−1.99	−1.99
(2.5, 0.0)	1.0	−0.09	−0.04	−0.02

$\kappa > 1$ throughout. The cubic restoring force x^3 far exceeds the linear drive $\mu x + a$, giving $\mathrm{Var}(r) \gg \mathrm{Var}(f)$ and large Cov.

Physical meaning: the Schlögl model is an overdamped nonequilibrium system. The restoring process not only absorbs the driving fluctuations but overshoots. This is the nonequilibrium analogue of overdamping in equilibrium thermodynamics.

Notably, as noise increases ($\sigma = 1.0$) near the bifurcation ($\mu = 2.5$), $\bar{\eta}$ rises from −2.0 to −0.02, approaching null regime. This likely reflects weakening restoring force as the linearised Jacobian's eigenvalue approaches zero.

5.3 η ≈ 0.20: a recurrent absorptive window

The Brusselator's $\eta \approx 0.19$–0.20 under its natural f/r decomposition coincides with values observed in entirely different domains:

System	Domain	f/r decomposition	η	Source
ZFCρ DP recursion	Number theory	Additive step / multiplicative screening	0.10–0.31 (median ≈ 0.20)	Thermo I
Lindley M/M/1	Queueing theory	Arrivals / service	≈ 0.15	Thermo I
Brusselator	Chemical kinetics	Synthesis / degradation	0.19–0.20	This paper

ZFCρ and Brusselator share no mathematical structure, physical domain, or dynamical equations. Their only common feature is that both are state-coupled, two-channel dissipative systems in which f and r share state variables.

We report this as a recurrent absorptive window in state-coupled two-channel systems. This is not a universality claim for η itself (the Schlögl $\bar{\eta} < 0$ and OU $\bar{\eta} \approx 1$ rule that out). It is a recurrent regime: in systems satisfying specific structural conditions (state-coupled, two-channel, restoring force not overwhelmingly dominant, noise not dominating the deterministic coupling), η tends to lie near 0.15–0.25.

Calibration of claim strength: $\eta \approx 0.20$ is the mean across the oscillatory regime (§5.4), not a constant. The full claim is: under moderate noise ($\sigma/x^* \leq 1$), on the canonical small-τ plateau of state-coupled two-channel oscillatory systems, η has a central tendency in [0.15, 0.25] and a parameter range in [0.14, 0.34]. When $\sigma/x^* > 1$, noise begins to disrupt deterministic coupling and η may exceed this range (e.g. $\eta = 0.43$ at $\sigma = 3.0$ in §5.2.2). Two state-coupled systems (ZFCρ, Brusselator) directly support this claim. Lindley M/M/1 ($\eta \approx 0.15$) is a channel-independent system whose η falls in a similar range possibly for different reasons, and is not counted as direct support for the state-coupled claim.

The phenomenon has a loose analogy with Feigenbaum's universality [8]: $\delta \approx 4.669$ appears in all period-doubling systems regardless of specific dynamics, arising from a renormalisation-group fixed point. Whether $\eta \approx 0.20$ similarly arises from a fixed point of f/r coupling dynamics is a high-priority open problem (§7.1).

5.4 Parameter-space scan: η as a continuous dynamical indicator

To test the robustness of $\eta \approx 0.20$, we scanned the Brusselator parameter space: $a \in \{0.5, 1.0, 1.5, 2.0\}$, b ranging from damped ($b < 1 + a^2$) into deep-oscillatory regime ($b \gg 1 + a^2$), $\sigma = 0.3$, $\tau = 0.05$.

Representative data points:

a	b	regime	η
0.5	0.75	damped	0.006
0.5	1.35	oscillatory	0.148
0.5	1.75	oscillatory	0.207
0.5	6.25	oscillatory	0.342
1.0	1.90	damped	0.124
1.0	2.10	oscillatory	0.140
1.0	3.00	oscillatory	0.206
1.0	7.00	oscillatory	0.260
2.0	4.90	damped	0.148
2.0	5.10	oscillatory	0.155
2.0	7.00	oscillatory	0.192

(Full 20-point dataset in supplementary material.)

Three key observations:

First, η varies continuously across the bifurcation $b = 1 + a^2$; there is no jump. Mean $\bar{\eta}$ in the damped regime is 0.103; mean η in the oscillatory regime is 0.203. The bifurcation is not a singular point of η but a smooth transition from low to moderate absorption.

Second, the oscillatory-regime range is [0.14, 0.34]. η is not constant; it increases monotonically with distance from bifurcation ($(b - b\_crit)/b\_crit$). $\eta \approx 0.20$ appears at moderate distance from bifurcation.

Third, across all a values, the oscillatory-regime mean η is near 0.20 (a = 0.5: 0.25; a = 1.0: 0.21; a = 1.5: 0.19; a = 2.0: 0.18). The influence of a is weaker than that of b.

η is therefore not a universal constant, but a continuous dynamical indicator. It increases with distance from bifurcation, taking the value ≈ 0.2 at moderate distance. This has more physical content than "η is always 0.2."

5.5 Composite dissipation structure: a candidate unifying ansatz

Thermo II (DOI: 10.5281/zenodo.19511064) showed $\eta \approx 0.20 = 1 - C(1)^{m_{eff}}$, where $C(1) \approx 0.96$ is the per-layer retention rate in ZFCρ and $m_{eff} \approx 5.47$ the effective layer count.

If this composite-dissipation mechanism is cross-system, the Brusselator's f-channel time series should also decompose into a similar exponential-decay structure. We computed the step-wise autocorrelation function on the $\tau = 0.05$ integrated f-channel series.

lag k	ACF_f(k)	ACF_f(k)/ACF_f(k−1)
0	1.000	—
1	0.982	0.982
2	0.938	0.956
3	0.883	0.942
5	0.770	—
10	0.525	—
15	0.324	—
20	0.189	—

The f-channel lag-1 autocorrelation is $C(1) = 0.982$. The ratio ACF(k)/ACF(k−1) is not strictly constant (declining from 0.982 to ~0.88), indicating that the Brusselator's f-channel autocorrelation decay is not purely exponential, though $C(1)^k$ serves as a first-order approximation.

Assuming $\eta = 1 - C(1)^{m_{eff}}$:

System	C(1)	m_eff	1 − C(1)^{m_eff}
ZFCρ	0.96	5.47	0.200
Brusselator f-channel	0.982	11.3	0.190

The individual parameters differ but the composite values are close. Different systems reach a similar η via different paths: ZFCρ via "96% retention per step, 5.5 steps per cycle," the Brusselator via "98% retention per step, 11 steps per cycle." Same product, different recipes.

Limitations of this analysis must be noted. In the Brusselator, $m_{eff} = 11.3$ is inferred from η and $C(1)$ ($m_{eff} = \ln(1 - \eta)/\ln(C(1))$), not independently measured. In ZFCρ, $m_{eff} = 5.47$ can be independently confirmed from the period of $G_j(M)$ damped oscillation, but no analogous independent measurement of m_{eff} exists for the Brusselator. The factorisation $1 - C(1)^{m_{eff}} \approx \eta$ is therefore a post hoc factorisation in the Brusselator, not an independent verification.

Furthermore, $C(1)$ in ZFCρ is a spatial autocorrelation across same-layer prime indices, while in the Brusselator it is the temporal lag-1 ACF of an integrated time series. Both can be described as "first-step retention," but they are not the same observable.

We report this as a candidate unifying ansatz, not an established mechanism. Its suggestive value lies in the observation that two completely different systems reach the same η via different ($C(1)$, m_{eff}) paths, hinting at a constraint that causes $1 - C(1)^{m_{eff}}$ to converge. This hint is suggestive, not conclusive.

5.6 Falsifiable predictions

Prediction P1 (parameter-space prediction). For any state-coupled two-channel oscillatory system under natural f/r decomposition and moderate noise ($\sigma/x^* \leq 1$), on the canonical small-τ plateau:

(i) η falls in [0.14, 0.34] across the oscillatory regime;

(ii) η increases monotonically with parametric distance from the Hopf bifurcation;

(iii) η varies continuously (no jump) at the bifurcation, transitioning smoothly from damped-regime low values ($\bar{\eta} < 0.15$) to oscillatory-regime moderate values ($\eta \approx 0.20$).

This can be tested immediately on the Van der Pol oscillator, Lotka–Volterra predator–prey model, FitzHugh–Nagumo neuron model, and similar standard systems. Any system whose natural-decomposition η falls outside [0.14, 0.34] constitutes a counterexample.

Prediction P2 (composite-dissipation ansatz). For any state-coupled system yielding $\eta \approx 0.2$, the f-channel (or r-channel) step-wise autocorrelation should exhibit approximate exponential decay $ACF(k) \approx C(1)^k$ such that $1 - C(1)^{m_{eff}} \approx \eta$.

$C(1)$ and m_{eff} vary with the system, but the composite $1 - C(1)^{m_{eff}}$ converges to [0.15, 0.25]. A system in which $ACF(k)$ decays as a power law rather than exponentially, or in which the composite value deviates from η by more than 50%, would constitute a counterexample.

Prediction P3 (experimental, hardest). In experimental data from the Belousov–Zhabotinsky reaction or its Oregonator model, if synthesis and degradation rates can be independently measured (via isotope labelling or rapid-mixing experiments), and under conditions where noise does not dominate the deterministic coupling and the natural f/r decomposition is identifiable, η should fall in the absorptive regime with a value in [0.14, 0.34].

This is a purely experimental prediction testable in a standard chemical-kinetics laboratory.

Prediction P4 ($\bar{\eta}(\tau)$ morphology predicts coupling type). Given an unknown system's time series and its f/r decomposition, the $\bar{\eta}(\tau)$ curve morphology diagnoses the f/r coupling type:

(i) If $\bar{\eta}(\tau)$ enters a plateau directly at small τ (no ascending segment), the system is state-coupled;

(ii) If $\bar{\eta}(\tau)$ descends from a high value ($\bar{\eta} \to 1$) to a plateau at small τ, the system is channel-independent.

This can be tested by blind measurement of $\bar{\eta}(\tau)$ on a new system of known coupling type.

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6. Interface with existing frameworks

6.1 Relation to three entropy-production decompositions

Nonequilibrium statistical mechanics has three mature decomposition frameworks. Onsager's force–flux decomposition [9] treats the near-equilibrium linear-response regime, writing entropy production as a product of generalised forces and fluxes. Hatano and Sasa's housekeeping/excess decomposition [10] splits entropy production into a steady-state maintenance cost and an excess from deviation. Esposito and Van den Broeck's adiabatic/non-adiabatic decomposition [11] further refines this partition.

The f/r decomposition is orthogonal to these three, and they are complementary. The key distinction is that f/r decomposes the dynamical processes themselves (which processes drive, which restore), while the others decompose entropy production (different sources of total dissipation). Both decompositions can be performed simultaneously: first extract η via f/r, then use Hatano–Sasa to trace which of η's physical sources are housekeeping contributions and which are excess.

κ and the off-diagonal elements of the Onsager matrix L share the feature of quantifying inter-process coupling strength. In the near-equilibrium linear-response regime, both measure "how much one process responds to the other." But κ is an operational definition requiring neither near-equilibrium conditions nor linearisation, while Onsager coefficients are products of linear-response theory. The two are not the same object.

6.2 Relation to information-theoretic quantities

$\kappa = \mathrm{Cov}(\Delta f, \Delta r) / \mathrm{Var}(\Delta f)$ is the regression slope β_{r~f} in a linear regression of Δr on Δf. Note that this is not the coefficient of determination R^2 (which equals $\mathrm{Corr}(\Delta f, \Delta r)^2$).

$\eta_{r\leftarrow f} = 1 - \kappa$ is therefore the directional linear-coupling residual under unit-response normalisation: if one unit of f-fluctuation is fully offset by one unit of r-response, then $\kappa = 1$ and $\eta = 0$. This is different from "how much of r's variance is explained by f" (the meaning of R^2). $\eta_{sym} = 1 - |Corr|$ is the symmetric coupling residual.

These quantities have a conceptual kinship with the Granger-causality literature [12]: Granger causality also measures how much one process's history helps predict the other. But $\eta_{r\leftarrow f}$ is synchronous (Cov within the same τ-window), while Granger causality is inter-temporal (past f predicting future r). They measure different aspects of inter-process dependence.

The mutual information in the Sagawa–Ueda information-thermodynamics framework [14] links measurement and system; the coupling measured by f/r decomposition links two internal processes. Both involve statistical dependence between two random variables, but in different physical contexts (external measurement vs internal coupling).

6.3 Complementarity with TUR

The thermodynamic uncertainty relation (TUR) gives $(\Delta J)^2/\langle J\rangle^2 \geqslant 2k_BT/\sigma_{tot}$ [4], an inequality constraining the precision–cost trade-off of a single current J.

η and TUR are complementary, not derivable from each other. TUR bounds current precision from below; η quantifies drive–response decoupling. TUR involves no two-process decomposition; it treats one net current.

Both move in the same qualitative direction: in systems with strong f/r symmetry breaking, the TUR bound is loose (high entropy production) and η is large (high decoupling). But this alignment is qualitative; no mathematical path derives η from TUR.

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7. Open problems

7.1 Structural explanation of η ≈ 0.20

ZFCρ and the Brusselator yield $\eta \approx 0.20$ under completely different dynamics. §5.5 showed this is not because C(1) or m_{eff} are individually constant, but because the composite $1 - C(1)^{m_{eff}}$ converges in both systems. ZFCρ's path: $C(1) = 0.96$, $m_{eff} = 5.47$. Brusselator's path: $C(1) = 0.982$, $m_{eff} = 11.3$.

The key open question: what mathematical structure (if any) guarantees convergence of $1 - C(1)^{m_{eff}}$ across different systems? One possible direction: view C(1) and m_{eff} as a point in a two-dimensional space and seek the contour on which $1 - C(1)^m = \text{const}$. Along this contour, higher C(1) (higher per-step retention) requires larger m_{eff} (more effective layers) to maintain the same composite dissipation. This is a form of compensation: systems can reach the same η via different (C(1), m_{eff}) combinations.

But this only describes the phenomenon; it does not explain why the contour is at $\eta \approx 0.2$ rather than at $\eta \approx 0.1$ or 0.5. Answering this may require starting from a fixed-point equation for f/r coupling. As noted in §5.5, m_eff in the Brusselator is currently inferred rather than independently measured; an independent measurement method is needed.

Additional systems are required. The current evidence—two state-coupled systems plus a parameter scan—is suggestive but not sufficient for a fully rigorous proof.

7.2 The specificity of natural decomposition

$\bar{\eta}$ is not a rotation invariant (§3.1). A natural conjecture was that the canonical decomposition maximises the $\bar{\eta}(\tau)$ plateau width (a maximum-scale-invariance principle).

We tested this numerically on the Brusselator. For each rotation angle θ (0° to 180°, step 10°), the $\bar{\eta}(\tau)$ plateau width (consecutive τ-points with < 10% variation) was computed over 25 log-spaced τ values. Note that this plateau analysis covers the full τ range, unlike the comparison-scale rotation experiment of §5.2.3 (which scans θ at a single τ). The natural decomposition yields $\eta \approx 0.20$ on the canonical small-τ plateau (§5.2.2) and $\bar{\eta} = 0.062$ at τ_relax (§5.2.3); these correspond to different τ scales.

Result: the conjecture fails. θ = 50°–100° give the widest plateaus (25/25 τ-points all stable), but $\bar{\eta} > 1$—outside the absorptive regime. Even among absorptive-regime decompositions, θ = 30° (width 16, $\bar{\eta} \approx 0.81$) is wider than θ = 0° (width 7, $\eta \approx 0.20$).

The natural decomposition has a different specificity: **it is the only decomposition whose η falls within the recurrent absorptive window (0.15–0.25).** θ = 10° gives $\bar{\eta} = 0.44$; θ = 20° gives $\bar{\eta} = 0.63$; θ = 30° gives $\bar{\eta} = 0.81$. Only θ = 0° (and the equivalent θ = 180°) gives $\eta \approx 0.20$.

This specificity connects to the candidate unifying ansatz of §5.5: only the natural decomposition produces an f-channel time series with the approximate exponential-decay autocorrelation structure that places 1 − C(1)^{m_eff} on the ≈ 0.20 contour. Rotation destroys this structure.

Thus the canonical status of the natural decomposition does not derive from an optimality principle (widest plateau or minimum η), but from its alignment with the underlying composite-dissipation structure. This is a deeper criterion than a variational principle, but harder to formalise. Its full understanding may require answering why 1 − C(1)^{m_eff} converges in state-coupled systems—the problem of §7.1.

7.3 Open quantum systems

The Lindblad master equation for open quantum systems, $d\rho/dt = -i[H, \rho] + \Sigma(L\rho L^\dagger - \dots)$, has a natural f/r decomposition: $f = -i[H, \rho]$ (unitary evolution) and $r = \Sigma(L\rho L^\dagger - \dots)$ (Lindblad dissipators). The decoherence time is a natural τ.

This direction currently lacks numerical verification. Whether $\bar{\eta}$ in a concrete open quantum system (e.g. a driven-dissipative cavity) falls in the absorptive regime is unknown.

7.4 η and Jacobian eigenstructure

Whether the Brusselator's $\eta \approx 0.19$ has a simple analytic relation to the Jacobian eigenvalues ($0.5 \pm 0.866i$) is an open question. The nonlinear term x^2y prevents complete linearisation from yielding a closed formula, but the correlation between η and the damping ratio $\mathrm{Re}(\lambda)/|\lambda|$ deserves further exploration.

7.5 Housekeeping entropy and η

In state-coupled systems, stronger coupling should imply higher housekeeping entropy production (the cost of maintaining f–r coordination) and lower η. This is a prediction testable in the Brusselator by varying parameters. If confirmed, it would provide a quantitative interface between η and the Hatano–Sasa framework.

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Appendix A: Protocol B (single-channel heuristic proxy)

When only a single time series $X(t)$ is available and f and r cannot be independently observed, an attractor-based conditional decomposition can be used.

Step 1: Estimate the attractor X^* (fixed point or limit-cycle centreline).

Step 2: Define deviation $\delta_n = X(n\tau) - X^*$.

Step 3: Conditional decomposition. If $|\delta_{n+1}| > |\delta_n|$ (moving away from attractor), assign ΔX_n to f. If $|\delta_{n+1}| < |\delta_n|$ (approaching attractor), assign to r.

Step 4: Compute κ and $\bar{\eta}$.

Limitations. Protocol B is a heuristic proxy, not an exact protocol. It assumes f and r alternate in dominance (rather than acting simultaneously); this fails for strongly coupled systems. $\bar{\eta}$ from Protocol B should not be regarded as an unbiased estimator of Protocol A's η.

In particular, for systems exhibiting limit cycles, the simple radial-distance criterion $|\delta_{n+1}| \gtrsim |\delta_n|$ fails, because the dynamics are predominantly tangential (phase evolution along the cycle) rather than radial (divergence/convergence). For such limit-cycle oscillators, Protocol B should be upgraded to phase-angle splitting in phase space, or one should rely on Protocol A's multi-variable equation-based extraction.

Theoretical note. Takens' embedding theorem [15] shows that a single time series with sufficient embedding dimension contains the topological information of the original dynamical system. Protocol B can be viewed as a first-order approximation to Takens reconstruction. However, the conditions for Takens' theorem (sufficient embedding dimension, low noise, dense sampling) are often violated in practical nonequilibrium systems.

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Appendix B: $\bar{\eta}(\tau)$ dual-morphology diagrams

(Formal version should include two figures.)

Figure B1: Channel-independent systems (Lindley M/M/1 type). $\bar{\eta}(\tau)$ exhibits three-region structure: under-resolved region ($\tau \ll \tau_{relax}$, $\bar{\eta}$ elevated) → canonical plateau → over-smoothed region ($\tau \gg \tau_{relax}$, $\bar{\eta}$ approaches asymptotic value).

Figure B2: State-coupled systems (Brusselator type). $\bar{\eta}(\tau)$ exhibits two-region structure: plateau (beginning at very small τ, $\eta \approx 0.19$) → monotone decline (large τ, tending toward 0 or negative). No under-resolved region.

The contrast between these two morphologies directly illustrates the core argument of §4.1: the shape of $\bar{\eta}(\tau)$ depends on the f/r coupling type, not on any universal property of η itself.